Chapter 1 Free-Response Practice Test

Directions: This practice test features free-response questions based on the content in Chapter 1: Limits and Continuity.

1.1: Defining a Limit
1.2: Evaluating Limits Analytically
1.3: Squeeze Theorem and Trigonometric Limits
1.4: Continuity
1.5: Formal Definition of a Limit
1.6: Limits with Infinity

For each question, show your work. If you encounter difficulties with a question, then move on and return to it later. Follow these guidelines:

- Do not use a calculator of any kind. All of these problems are designed to contain simple numbers.
- Adhere to the time limit of 90 minutes.
- After you complete all the questions, score yourself according to the Solutions document. Note any topics that require revision.

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Limits and Continuity

Number of Questions—20

Suggested Time—1 hour 30 minutes

NO CALCULATOR

Scoring Chart

Section	Points Earned	Points Available
Rapid Limits		20
Short Questions		35
Question 18		15
Question 19		15
Question 20		15
TOTAL		**100**

Rapid Limits

Evaluate the limit if it exists.

1. $\lim_{x\to\infty} \frac{2}{x^6}$ *(2 pts.)*

2. $\lim_{x\to 3} \sqrt{x^3+x^2-8}$ *(2 pts.)*

3. $\displaystyle\lim_{x\to-\infty}\frac{5-x^3}{8x^4+x^2}$ *(2 pts.)*

4. $\displaystyle\lim_{x\to 4}\frac{8x-2x^2}{x^4-16x^2}$ *(2 pts.)*

5. $\lim\limits_{x \to 2} \cos\left(\frac{1}{x-2}\right)$ *(2 pts.)*

6. $\lim\limits_{x \to 0} \frac{5 - 5\cos x}{10x}$ *(2 pts.)*

7. $\displaystyle\lim_{x\to\infty}\frac{e^x}{4e^x+e^{2x}}$ *(2 pts.)*

8. $\displaystyle\lim_{\theta\to\pi/2}\frac{\sin 2\theta}{9-9\cos^2\theta}$ *(2 pts.)*

9. $\lim_{x\to 3} \dfrac{3x-9}{\sin(2x-6)}$ *(2 pts.)*

10. $\lim_{x\to 4} \dfrac{\sqrt{4x-7}-3}{8-2x}$ *(2 pts.)*

Short Questions

11. If $\lim_{x \to -2} f(x) = 6$ and $\lim_{x \to -2} g(x) = 8$, then find $\lim_{x \to -2}[g(x) - f(x)]$ and $\lim_{x \to -2} \dfrac{f(x)+2}{3g(x)}$. *(5 pts.)*

12. What type of discontinuity, if any, does $f(x) = \dfrac{6-2x}{9-x^2}$ have at $x = 3$? *(5 pts.)*

13. Determine the interval of continuity of $g(x) = \dfrac{\sqrt{8-x}}{2x^3-8x}$. *(5 pts.)*

14. Find all the asymptotes of the rational function $f(x) = \dfrac{2x^2-5x+3}{x^2+x-6}$. *(5 pts.)*

15. If $\lim_{x\to 0^-} f(x) = 4$ and $\lim_{x\to 0^+} f(x) = 7$, then evaluate $\lim_{x\to 3^+} f(9-x^2)$ if it exists. *(5 pts.)*

16. Show that $\lim_{x\to 0}\left[6+\frac{1}{2}x\cos\left(\frac{3}{x}\right)\right] = 6$. *(5 pts.)*

17. Construct a proof for $\lim_{x \to 5^+} \ln(x-5) = -\infty$. *(5 pts.)*

Long Questions

18. Let $f(x) = x^3 - 3x^2 + x + 1$ and $g(x) = \dfrac{x^2 - 4x + 3}{x - 1}$. The graphs of both functions intersect at the point $(2, -1)$. Note that $f(-1) = -4$ and $f(3) = 4$.

(a) Does the graph of $y = g(x)$ contain a removable discontinuity? If so, then find the discontinuity's coordinates. *(3 pts.)*

(b) Justify why f must have at least one zero on $(-1, 3)$. *(3 pts.)*

(c) Find $\lim_{x \to 1} f(g(x))$ and $\lim_{x \to 0} g(f(x))$. Show the work that leads to your answers. *(5 pts.)*

(d) Let h be a function that satisfies $g(x) \leqslant h(x) \leqslant f(x)$ for all $x \geqslant -1$. Show that $h(x)$ is continuous at $x = 2$. *(4 pts.)*

19. The population of cows on a farm is modeled by the continuous function $N(t)$ for $t \geqslant 0$, where t is measured in years. It is known that $N(0) = 2000$ and $N(3) = 2500$.

(a) Interpret the meaning of $\lim_{t\to\infty} N(t) = 2700$ in context. *(1 pt.)*

(b) Justify why there exists a value c in $(0, 3)$ for which $N(c) = 2200$. *(2 pts.)*

(c) Around $t = 5$, amid a shortage of resources, it is observed that $N(t) = t^2 - 4t + 2100$. *(6 pts.)*

At that time, the population grows at a rate given by $\lim_{t\to 5} \dfrac{N(t) - N(5)}{t - 5}$. Find this rate of change.

(d) For $t \neq 7$, the growth rate of the cow population is modeled by *(6 pts.)*

$r(t) = \dfrac{t^2 - 10t + 21}{3 - t} \sin\left(\dfrac{1}{t-7}\right)$. Find, and interpret in context, the value of $\lim\limits_{t \to 7} r(t)$ using the Squeeze Theorem.

20. Let f be a piecewise function defined by

$$f(x) = \begin{cases} 2+\sin\left(\dfrac{\pi}{2}x\right) & x \leqslant 1 \\ px+q & 1 < x \leqslant 2\,. \end{cases}$$

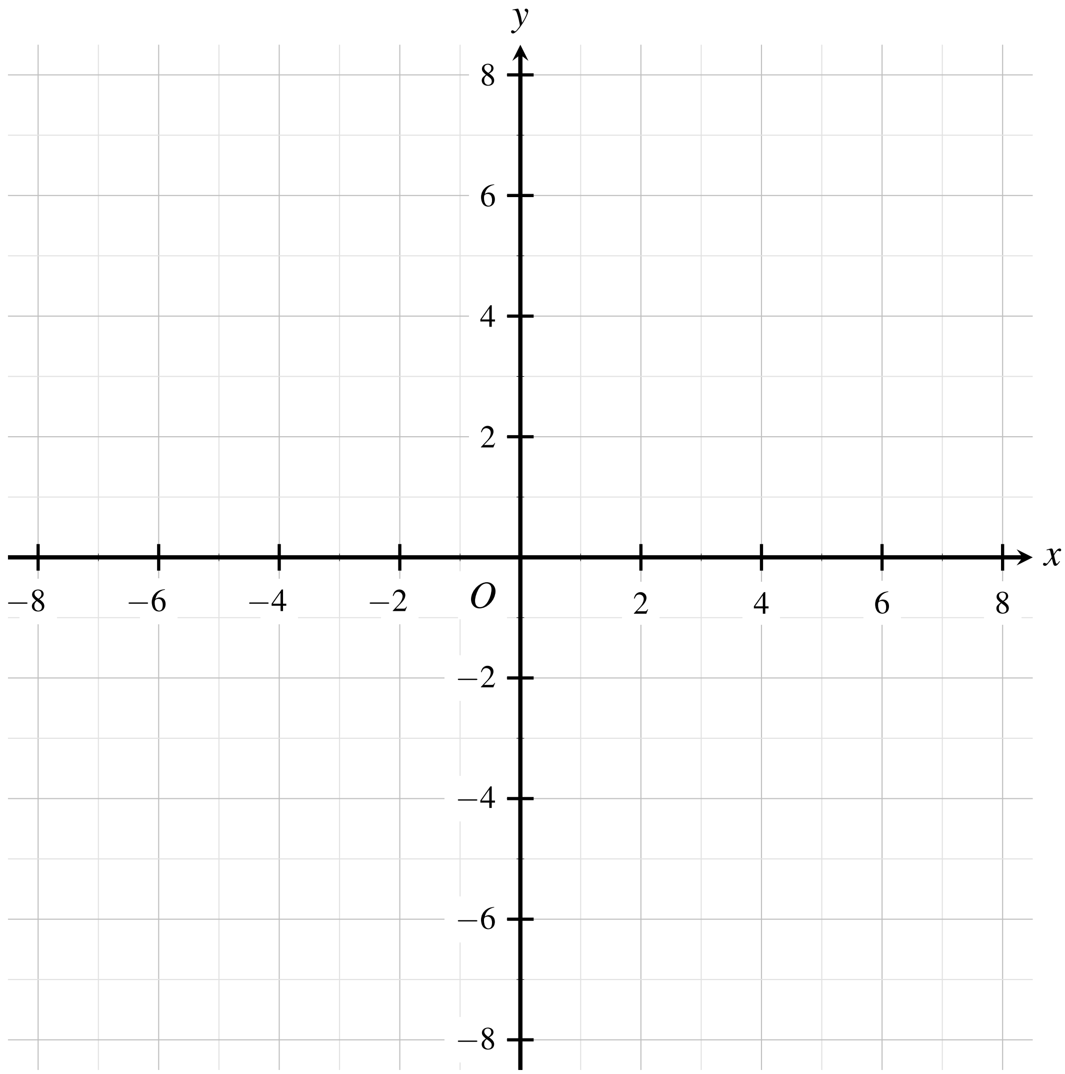
It is known that $f(2) = 4$ and f is continuous at 1.

(a) Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1^-} f(x)$. *(2 pts.)*

(b) What are the values of p and q ? *(3 pts.)*

(c) Using the values of p and q from part (b), prove that $\lim_{x\to 1}(px+q) = p+q$ using the Epsilon-Delta definition of a limit. *(5 pts.)*

(d) It is known that $\lim_{x\to 4^-} f(x) = \infty$, $\lim_{x\to 4^+} f(x) = -\infty$, and $\lim_{x\to\infty} f(x) = 6$. Sketch a possible graph of $y = f(x)$. *(5 pts.)*



This marks the end of the test. The solutions and scoring rubric begin on the next page.

Rapid Limits (2 points each)

1. As $x \to \infty$, the denominator increases, decreasing the fraction's value. Thus,

$$\lim_{x\to\infty} \frac{2}{x^6} = \boxed{0}$$

**

2. By Direct Substitution,

$$\begin{aligned}\lim_{x\to 3} \sqrt{x^3+x^2-8} &= \sqrt{(3)^3+(3)^2-8} \\ &= \boxed{\sqrt{28}}\end{aligned}$$

**

3. This is the limit at infinity of a rational expression, so the highest-order terms govern:

$$\begin{aligned}\lim_{x\to-\infty} \frac{5-x^3}{8x^4+x^2} &= \lim_{x\to-\infty} \frac{-x^3}{8x^4} \\ &= \lim_{x\to-\infty} \frac{-1}{8x} \\ &= \boxed{0}\end{aligned}$$

**

4. This limit is in the indeterminate form $\frac{0}{0}$. Factoring the rational expression and canceling a common factor, we see

$$\begin{aligned}
\lim_{x\to 4}\frac{8x-2x^2}{x^4-16x^2} &= \lim_{x\to 4}\frac{2x(4-x)}{x^2(x+4)(x-4)}\\
&= \lim_{x\to 4}\frac{2x\cancel{(4-x)}}{-x^2(x+4)\cancel{(4-x)}}\\
&= \lim_{x\to 4}\frac{2x}{-x^2(x+4)}\\
&= \frac{2(4)}{-(4)^2(4+4)}\\
&= \boxed{-\frac{1}{16}}
\end{aligned}$$

**

5. As $x\to 2$, $\cos\left(\frac{1}{x-2}\right)$ oscillates continually and never settles to a final value. Thus,

$$\lim_{x\to 2}\cos\left(\frac{1}{x-2}\right)\ \boxed{\text{does not exist}}$$

**

6. We have

$$\begin{aligned}
\lim_{x\to 0}\frac{5-5\cos x}{10x} &= \lim_{x\to 0}\frac{5(1-\cos x)}{10x}\\
&= \frac{1}{2}\lim_{x\to 0}\frac{1-\cos x}{x}\\
&= \frac{1}{2}(0)\\
&= \boxed{0}
\end{aligned}$$

**

7. Dividing the numerator and denominator each by e^x shows

$$\lim_{x\to\infty}\frac{e^x}{4e^x+e^{2x}}=\lim_{x\to\infty}\frac{1}{4+e^x}$$

$$=\boxed{0}$$ **

8. Using trigonometric identities (the double-angle identity for sine and the Pythagorean identity) shows

$$\lim_{\theta\to\pi/2}\frac{\sin 2\theta}{9-9\cos^2\theta}=\lim_{\theta\to\pi/2}\frac{2\sin\theta\cos\theta}{9(1-\cos^2\theta)}$$

$$=\lim_{\theta\to\pi/2}\frac{2\sin\theta\cos\theta}{9\sin^2\theta}$$

$$=\lim_{\theta\to\pi/2}\frac{2\cos\theta}{9\sin\theta}$$

$$=\frac{2\cos\left(\dfrac{\pi}{2}\right)}{9\sin\left(\dfrac{\pi}{2}\right)}=\boxed{0}$$ **

9. We write

$$\lim_{x\to 3}\frac{3x-9}{\sin(2x-6)}=\lim_{x\to 3}\frac{\frac{3}{2}(2x-6)}{\sin(2x-6)}.$$

We let $t=2x-6$. As $x\to 3$, $t\to 0$. So the limit becomes

$$\lim_{t\to 0}\frac{\frac{3}{2}t}{\sin t}=\frac{3}{2}\lim_{t\to 0}\frac{1}{(\sin t)/t}$$

$$=\tfrac{3}{2}(1)$$

$$=\boxed{\frac{3}{2}}$$ **

10. Multiplying the numerator and denominator each by the conjugate of the radical expression—that is,

$\sqrt{4x-7}+3$—gives

$$\begin{aligned}
\lim_{x\to4}\frac{\sqrt{4x-7}-3}{8-2x}\cdot\frac{\sqrt{4x-7}+3}{\sqrt{4x-7}+3} &= \lim_{x\to4}\frac{(4x-7)-9}{(8-2x)(\sqrt{4x-7}+3)}\\
&= \lim_{x\to4}\frac{4x-16}{(8-2x)(\sqrt{4x-7}+3)}\\
&= \lim_{x\to4}\frac{4\cancel{(x-4)}}{-2\cancel{(x-4)}(\sqrt{4x-7}+3)}\\
&= \lim_{x\to4}\frac{4}{-2(\sqrt{4x-7}+3)}\\
&= \frac{4}{-2(\sqrt{4(4)-7}+3)}\\
&= \boxed{-\frac{1}{3}}
\end{aligned}$$

**

Short Questions (5 points each)

11. By Limit Laws,

$$\begin{aligned}\lim_{x\to-2}[g(x)-f(x)] &= \lim_{x\to-2} g(x) - \lim_{x\to-2} f(x)\\ &= 8-6\\ &= \boxed{2}\end{aligned}$$ **

In addition,

$$\begin{aligned}\lim_{x\to-2}\frac{f(x)+2}{3g(x)} &= \frac{\lim_{x\to-2}[f(x)+2]}{3\lim_{x\to-2} g(x)} \quad * \\ &= \frac{6+2}{3(8)}\\ &= \boxed{\frac{1}{3}}\end{aligned}$$ **

12. Factoring the function shows

$$f(x)=\frac{2\cancel{(3-x)}}{(3+x)\cancel{(3-x)}}=\frac{2}{3+x}, \quad x\neq 3.$$ **

In addition,

$$\lim_{x\to3} f(x) = \lim_{x\to3}\frac{2}{3+3} = \frac{1}{3}.$$ *

Because the limit exists as $x\to3$, it follows that at $x=3$, f has a

$$\boxed{\text{removable discontinuity}}$$ **

13. Any combination of a radical function and a polynomial is continuous on its domain. The numerator is defined when the argument of the square root is nonnegative—that is,

$$8-x\geqslant 0 \implies x\leqslant 8.$$ *

In addition, the denominator must be nonzero. The denominator is 0 when

$$2x^3 - 8x = 0 \quad *$$

$$2x(x+2)(x-2) = 0$$

$$\implies x = -2, 0, 2. \quad *$$

The interval of continuity contains all $x \leqslant 8$ with $x \neq -2, 0, 2$—namely,

$$\boxed{(-\infty, -2) \cup (-2, 0) \cup (0, 2) \cup (2, 8]} \quad **$$

14. The function f has vertical asymptotes when the denominator is 0 and the numerator is nonzero. Solving, we see

$$x^2 + x - 6 = 0$$

$$(x+3)(x-2) = 0$$

$$\implies x = -3, 2. \quad *$$

The numerator is nonzero at both values, so $x = -3$ and $x = 2$ are both vertical asymptotes. In addition,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2}{x^2} = 2, \quad *$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x^2}{x^2} = 2. \quad *$$

Hence, the asymptotes are as follows:

horizontal asymptotes: $\boxed{y = 2}$ *

vertical asymptotes: $\boxed{x = -3}$ $\boxed{x = 2}$ *

15. Let $t = 9 - x^2$. As $x \to 3^+$, $t \to 0^-$ (meaning $t = 9 - x^2$ only takes on values less than 0 as $x \to 3^+$). Thus,

$$\lim_{x\to 3^+} f(9-x^2) = \lim_{t\to 0^-} f(t) \qquad ***$$

$$= \boxed{4} \qquad **$$

16. The range of cosine is $[-1,1]$, so

$$-1 \leqslant \cos\left(\frac{3}{x}\right) \leqslant 1\,. \qquad *$$

Because $-|x| \leqslant x \leqslant |x|$, we have

$$6 - \frac{1}{2}|x| \leqslant 6 + \frac{1}{2}x\cos\left(\frac{3}{x}\right) \leqslant 6 + \frac{1}{2}|x|\,. \qquad *$$

Note that

$$\lim_{x\to 0}\left(6 - \frac{1}{2}|x|\right) = \lim_{x\to 0}\left(6 + \frac{1}{2}|x|\right) = 6\,. \qquad *$$

Thus, by the Squeeze Theorem, it also follows that

$$\lim_{x\to 0}\left[6 + \frac{1}{2}x\cos\left(\frac{3}{x}\right)\right] = \boxed{6} \qquad **$$

17. For every $M < 0$, there must exist a number $\delta > 0$ such that

$$\ln(x-5) < M \qquad \text{if} \qquad 0 < x-5 < \delta\,. \qquad *$$

From the first inequality, we attain

$$x - 5 < e^M\,. \qquad *$$

Comparing this result to $0 < x-5 < \delta$, we can choose

$$\delta = e^M\,. \qquad **$$

Thus, with $\delta = e^M$, it follows that for $0 < x-5 < \delta$,

$$\ln(x-5) < M \qquad \text{if} \qquad 0 < x-5 < e^M\,, \qquad *$$

proving that $\lim_{x \to 5^+} \ln(x-5) = -\infty$.

Long Questions (15 points each)

18. (a) Factoring the function g shows

$$g(x) = \frac{(x-3)\cancel{(x-1)}}{\cancel{(x-1)}} = x-3\,, \quad x \neq 1\,.$$ *

There is therefore a removable discontinuity at $x = 1$. The y-coordinate of this discontinuity is

$$\lim_{x\to 1} g(x) = \lim_{x\to 1}(x-3) = -2\,.$$ *

Hence, the removable discontinuity is located at

$$\boxed{(1,-2)}$$ *

(b) Because f is a polynomial, it is continuous on $\mathbb{R}$, including on $[-1,3]$. Thus, by the Intermediate Value Theorem, f takes on every value between $f(-1) = -4$ and $f(3) = 4$ on $[-1,3]$, which includes 0. Accordingly, f must have at least one zero on $(-1,3)$. ***

(c) Note that

$$\lim_{x\to 1} g(x) = \lim_{x\to 1}(x-3) = -2\,.$$ *

Because f is continuous,

$$\lim_{x\to 1} f(g(x)) = f\left(\lim_{x\to 1} g(x)\right) = f(-2) = \boxed{-21}$$ *

In addition, note that

$$\lim_{x\to 0} f(x) = 1\,.$$ *

Thus, with $t = f(x)$, it follows that $t \to 1$ as $x \to 0$. Accordingly,

$$\lim_{x\to 0} g(f(x)) = \lim_{t\to 1} g(t)$$ *

$$= \boxed{-2}$$ *

(d) For continuity, we must show that

$$\lim_{x\to 2} h(x) = h(2)\,.$$

First, we have

$$g(2) = f(2) = -1\,.$$

Because $g(x) \leqslant h(x) \leqslant f(x)$ holds for all $x \geqslant -1$, we must have $g(2) \leqslant h(2) \leqslant f(2) \iff -1 \leqslant h(2) \leqslant -1$ and thus

$$h(2) = -1\,.$$ *

Moreover,

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = -1\,.$$ *

By the Squeeze Theorem, it also follows that

$$\lim_{x\to 2} h(x) = -1\,.$$ *

Because

$$\lim_{x\to 2} h(x) = h(2) = -1\,,$$ *

h is continuous at $x = 2$.

19. (a) The population of cows approaches a limit of 2700 over a very long time. *

(b) Because $N(t)$ is continuous on $[0,3]$, the Intermediate Value Theorem asserts that N takes on every value between $N(0) = 2000$ and $N(3) = 2500$ on $[0,3]$. Thus, N must take on the value 2200 at least once in $(0,3)$, so a value c exists in $(0,3)$ such that $N(c) = 2200$. **

(c) Note that $N(5) = 2105$. We then have

$$\begin{aligned}
\lim_{t\to 5} \frac{N(t)-N(5)}{t-5} &= \lim_{t\to 5} \frac{(t^2-4t+2100)-2105}{t-5} && * \\
&= \lim_{t\to 5} \frac{t^2-4t-5}{t-5} && * \\
&= \lim_{t\to 5} \frac{\cancel{(t-5)}(t+1)}{\cancel{(t-5)}} && * \\
&= \lim_{t\to 5} (t+1) && * \\
&= \boxed{6 \text{ cows/yr}} && **
\end{aligned}$$

(d) The range of sine is $[-1,1]$, so we have

$$-1 \leqslant \sin\left(\frac{1}{t-7}\right) \leqslant 1\,.$$

We also have

$$-\left|\frac{t^2-10t+21}{3-t}\right| \leqslant \frac{t^2-10t+21}{3-t} \leqslant \left|\frac{t^2-10t+21}{3-t}\right|\,.$$

Thus,

$$-\left|\frac{t^2-10t+21}{t-3}\right| \leqslant \frac{t^2-10t+21}{3-t}\sin\left(\frac{1}{t-7}\right) \leqslant \left|\frac{t^2-10t+21}{3-t}\right|\,.$$

Note that

$$\lim_{t\to 7}\left(-\left|\frac{t^2-10t+21}{t-3}\right|\right) = \lim_{t\to 7}\left(-\left|\frac{(t-7)\cancel{(t-3)}}{\cancel{(t-3)}}\right|\right) = 0\,,$$

$$\lim_{t\to 7}\left|\frac{t^2-10t+21}{3-t}\right| = \lim_{t\to 7}\left|\frac{(t-7)\cancel{(t-3)}}{-\cancel{(t-3)}}\right| = 0\,.$$

The Squeeze Theorem also gives

$$\lim_{t\to 7}\frac{t^2-10t+21}{3-t}\sin\left(\frac{1}{t-7}\right) = \boxed{0\text{ cows/yr}}$$

Hence, near $t=7$ the cow population becomes stagnant, with the growth rate approaching zero.

20. (a) The limits are

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[2+\sin\left(\frac{\pi}{2}x\right)\right] = 2+\sin 0 = \boxed{2}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\left[2+\sin\left(\frac{\pi}{2}x\right)\right] = 2+\sin\left(\frac{\pi}{2}\right) = \boxed{3}$$

(b) For continuity at 1, we need $\lim\limits_{x\to 1} f(x) = f(1)$. The one-sided limits must be equal:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$

$$3 = p+q\,.$$

In addition, since $f(2)=4$, substituting $x=2$ gives

$$2p+q=4. \quad *$$

The solution to the system of equations $3=p+q$ and $2p+q=4$ is

$$\boxed{p=1} \qquad \boxed{q=2} \quad *$$

(c) We must show that $\lim\limits_{x\to 1}(x+2)=3$. For every number $\varepsilon>0$, there must exist a number $\delta>0$ such that

$$|(x+2)-3|<\varepsilon \qquad \text{if} \qquad 0<|x-1|<\delta. \quad *$$

From the first inequality, we have

$$|x-1|<\varepsilon. \quad *$$

Comparing this inequality to $0<|x-1|<\delta$, we can choose

$$\delta=\varepsilon. \quad *$$

Then for $0<|x-1|<\delta$,

$$|(x+2)-3|=|x-1|<\delta=\varepsilon. \quad *$$

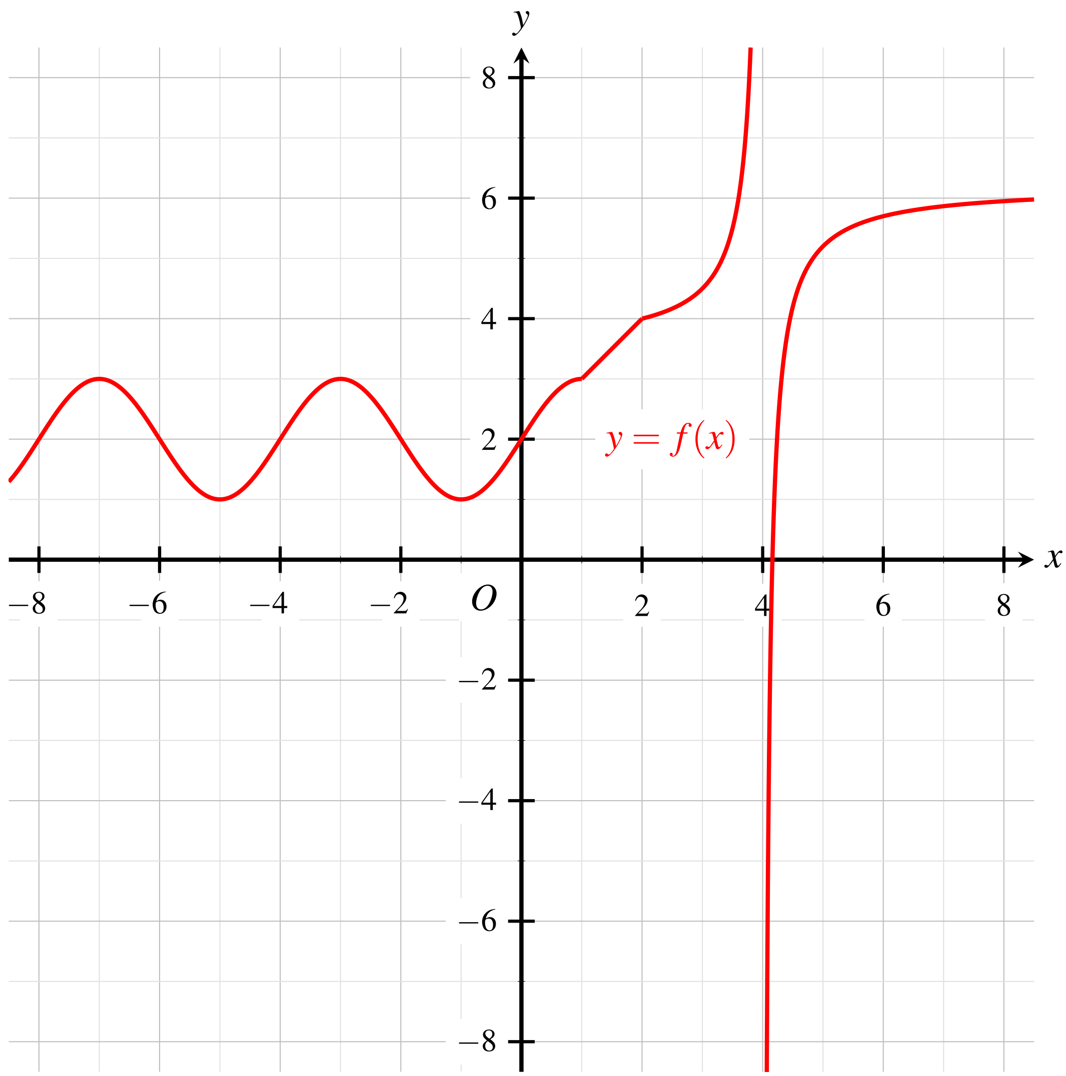
Accordingly,

$$|(x+2)-3|<\varepsilon \qquad \text{if} \qquad 0<|x-1|<\varepsilon. \quad *$$

This proves that $\lim\limits_{x\to 1}(x+2)=3$.

(d) The graph of $y=f(x)$ has the following features:

- $f(x)=2+\sin\left(\dfrac{\pi}{2}x\right)$ for $x\leqslant 1$ *
- $f(x)=x+2$ from $x=1$ to $x=2$ *
- Horizontal asymptote of $y=6$ *
- Vertical asymptote of $x=4$ *
- $\lim\limits_{x\to 4^-} f(x)=\infty$ and $\lim\limits_{x\to 4^+} f(x)=-\infty$ *


y
x
8
6
4
2
−2
−4
−6
−8
O
$y = f(x)$