Chapter 10 Free-Response Practice Test

Directions: This practice test features free-response questions based on the content in Chapter 10: Infinite Sequences and Series.

10.1: Sequences
10.2: Infinite Series and Divergence Test
10.3: Integral Test
10.4: Comparison Tests
10.5: Alternating Series
10.6: Absolute Convergence and the Ratio and Root Tests
10.7: Power Series
10.8: Taylor and Maclaurin Series

For each question, show your work. If you encounter difficulties with a question, then move on and return to it later. Follow these guidelines:

- Do not use a calculator of any kind. All of these problems are designed to contain simple numbers.
- Adhere to the time limit of 90 minutes.
- After you complete all the questions, score yourself according to the Solutions document. Note any topics that require revision.

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Infinite Sequences and Series

Number of Questions—20

Suggested Time—1 hour 30 minutes

NO CALCULATOR

Scoring Chart

Section	Points	Points Available
Rapid Series		20
Short Questions		35
Question 18		15
Question 19		15
Question 20		15
TOTAL		**100**

Rapid Series

Determine whether the series converges or diverges. No partial credit is awarded for an incorrect conclusion.

1. $\sum_{n=1}^{\infty} 2$ *(2 pts.)*

2. $\sum_{n=0}^{\infty} 5\left(\frac{1}{4}\right)^n$ *(2 pts.)*

3. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+11}}$ (2 pts.)

4. $\displaystyle\sum_{n=1}^{\infty} \frac{4n^2-n+3}{5n^3-7n^2+4}$ (2 pts.)

5. $\displaystyle\sum_{n=3}^{\infty} \frac{e^{7n}}{n!}$ (2 pts.)

6. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ (2 pts.)

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^4}{3n^4+8}$ *(2 pts.)*

8. $\displaystyle\sum_{n=9}^{\infty} \frac{\cos(\pi n)}{8-n}$ *(2 pts.)*

9. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+3}{8n+6}\right)^{2n}$ (2 pts.)

10. $\displaystyle\sum_{n=1}^{\infty}\frac{\sin^3 n}{\sqrt{n^3}}$ (2 pts.)

Short Questions

11. Calculate the exact value of $\sum_{n=0}^{\infty} \frac{2}{n^2+8n+15}$. (5 pts.)

12. Write the first three terms of the Maclaurin series of $\sqrt[5]{4+x}$. (5 pts.)

13. Based on the Alternating Series Error Bound, find the smallest integer N such that $S_N = \sum_{i=1}^{N} \frac{(-1)^i}{(i+4)^3}$ approximates $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+4)^3}$ with an error of less than 0.001. *(5 pts.)*

14. Find the third-degree Taylor polynomial for $f(x) = \cos 2x$ centered at $x = \frac{\pi}{3}$. *(5 pts.)*

15. Write a Maclaurin series for $\ln\left(1+\frac{x}{2}\right)$ and determine its interval of convergence. (*5 pts.*)

16. Based on the Integral Test, what is the maximum error with which $S_5 = \sum_{i=1}^{5} \frac{1}{(i+5)^3}$ approximates $S = \sum_{n=1}^{\infty} \frac{1}{(n+5)^3}$? Verify all the conditions of the Integral Test. (*5 pts.*)

17. Find a power series solution for the initial value problem $y' = \dfrac{2x^3}{x^7+1}$, $y(0) = 5$. *(5 pts.)*

Long Questions

18. Consider the sequence $\{a_n\}_1^\infty = \left\{2 + \frac{\cos n}{n}\right\}_1^\infty$.

(a) Is $\{a_n\}_1^\infty$ bounded? Is it monotonic? (3 pts.)

(b) Find the limit of $\{a_n\}_1^\infty$ or show that it diverges. (4 pts.)

(c) Let $b_n = \frac{n^2+4n+6}{2n^2+7}$. What is the limit of the sequence $\left\{\frac{2a_n}{b_n}\right\}$? (4 pts.)

(d) Another sequence is defined by $\{c_n\}_1^\infty = \left\{-\frac{1}{4}, \frac{2}{5}, -\frac{4}{6}, \frac{8}{7}, \ldots\right\}$. Determine c_n. *(4 pts.)*

19. A function f has the power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+4} x^n$.

(a) Determine the interval of convergence of the power series. (6 pts.)

(b) What power series represents $f(2x)$? Find its interval of convergence. (3 pts.)

(c) Find a power series for $f'(x)$. (2 pts.)

(d) Let $g(x) = \int_0^x f(t)\,\mathrm{d}t$. Write a power series for g. *(4 pts.)*

20. The Taylor series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{4^n}{n}(x-5)^n$ converges to $f(x)$ for $|x-5|<R$, where R is the radius of convergence.

(a) Determine R. (3 pts.)

(b) Write the third-degree Taylor polynomial for f and use it to approximate $f(6)$. (2 pts.)

(c) Given $\left|f^{(4)}(x)\right| \leqslant 40$ for all $5 \leqslant x \leqslant 6$, use Taylor's Remainder Theorem to determine the magnitude of the maximum error in the estimate from part (b). (3 pts.)

(d) Find $f'(5)$ and $f''(5)$. *(3 pts.)*

(e) A fourth-degree Maclaurin series for f is $4 - \frac{x^2}{5} + 2x^3 - \frac{x^4}{10}$. Let $g(x) = f(x)\sin x$. *(4 pts.)*

Write the third-degree Maclaurin polynomial for g.

This marks the end of the test. The solutions and scoring rubric begin on the next page.

Rapid Series (2 points each)

1. Note that $\lim_{n\to\infty} 2 = 2 \neq 0$. By the Divergence Test, the series $\boxed{\text{diverges}}$. **

2. The series is geometric with common ratio $r = 1/4$. Since $-1 < r < 1$, the series $\boxed{\text{converges}}$. **

3. The series is equivalent to $\sum_{n=11}^{\infty} \frac{1}{\sqrt{n}}$, which is a p-series with $p = 1/2$. Since $p \leqslant 1$, the series $\boxed{\text{diverges}}$. The Integral Test or a comparison test can prove the same result. **

4. The series with the highest-degree terms is

$$\sum_{n=1}^{\infty} \frac{4n^2}{5n^3} = \sum_{n=1}^{\infty} \frac{4}{5n} = \frac{4}{5} \sum_{n=1}^{\infty} \frac{1}{n},$$

the divergent Harmonic series. Then we compare $a_n = \frac{4n^2 - n + 3}{5n^3 - 7n^2 + 4}$ to $b_n = \frac{1}{n}$ and take the limit of their ratio:

$$\begin{aligned} L = \lim_{n\to\infty} \frac{\frac{4n^2 - n + 3}{5n^3 - 7n^2 + 4}}{1/n} \\ &= \lim_{n\to\infty} \frac{4n^3 - n^2 + 3n}{5n^3 - 7n^2 + 4} \\ &= \lim_{n\to\infty} \frac{4n^3}{5n^3} \\ &= \frac{4}{5}. \end{aligned}$$

Because L is positive and finite, the Limit Comparison Test asserts that $\sum_{n=1}^{\infty} a_n$ also $\boxed{\text{diverges}}$. **

5. By applying the Ratio Test,

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{e^{7(n+1)}}{(n+1)!}\cdot\frac{n!}{e^{7n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{e^{7n}e^{7}}{(n+1)\,n!}\cdot\frac{n!}{e^{7n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{e^{7}}{n+1}\right| \\ &= e^{7}\lim_{n\to\infty}\frac{1}{n+1} \\ &= 0\,. \end{aligned}$$

Because $L < 1$, the series $\boxed{\text{converges}}$. **

6. Note that $n^2\ln n > n^2$ for sufficiently large n, so

$$0 < \frac{1}{n^2\ln n} < \frac{1}{n^2}\,.$$

Also, $\sum_{n=2}^{\infty}\frac{1}{n^2}$ converges $(p = 2 > 1)$, so by the Direct Comparison Test $\sum_{n=2}^{\infty}\frac{1}{n^2\ln n}$ also $\boxed{\text{converges}}$. **

7. The series is alternating. But

$$\lim_{n\to\infty}\frac{n^4}{3n^4+8} = \frac{1}{3} \neq 0\,,$$

so the Alternating Series Test cannot be used. Instead, observe that $\lim_{n\to\infty}\frac{(-1)^n n^4}{3n^4+8}$ does not exist; as n grows larger, the function's values alternate between $-1/3$ and $1/3$. Hence, by the Divergence Test the series $\boxed{\text{diverges}}$. **

8. Note that $\cos(\pi n) = (-1)^n$, so the series is alternating. The series is equivalent to $\sum_{n=9}^{\infty}\frac{(-1)^{n+1}}{n-8}$. We see $\frac{1}{n-8}$ is decreasing for all $n \geqslant 9$ as well as $\lim_{n\to\infty}\frac{1}{n-8} = 0$. By the Alternating Series Test, the series $\boxed{\text{converges}}$. **

9. Applying the Root Test shows

$$L = \lim_{n\to\infty}\left[\left(\frac{2n+3}{8n+6}\right)^{2n}\right]^{1/n}$$

$$= \lim_{n\to\infty}\left(\frac{2n+3}{8n+6}\right)^{2}$$

$$= \left(\frac{2}{8}\right)^{2}$$

$$= \frac{1}{16}.$$

Because $L < 1$, the series $\boxed{\text{converges}}$. **

10. Because $0 \leqslant \left|\sin^3 n\right| \leqslant 1$,

$$0 \leqslant \frac{\left|\sin^3 n\right|}{\sqrt{n^3}} \leqslant \frac{1}{\sqrt{n^3}}.$$

Because $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3}}$ converges ($p = \frac{3}{2} > 1$), the Direct Comparison Test holds that $\sum_{n=1}^{\infty}\frac{\left|\sin^3 n\right|}{\sqrt{n^3}}$ converges. Thus, $\sum_{n=1}^{\infty}\frac{\sin^3 n}{\sqrt{n^3}}$ is absolutely convergent, so it $\boxed{\text{converges}}$. **

Short Questions (5 points each)

11. We split the fraction into partial fractions, attaining

$$\sum_{n=0}^{\infty}\left(\frac{-1}{n+5}+\frac{1}{n+3}\right).$$ **

Writing out terms shows

$$\sum_{n=0}^{\infty}\left(\frac{-1}{n+5}+\frac{1}{n+3}\right)=\left(-\cancel{\frac{1}{5}}+\frac{1}{3}\right)+\left(-\cancel{\frac{1}{6}}+\frac{1}{4}\right)+\left(-\cancel{\frac{1}{7}}+\cancel{\frac{1}{5}}\right)$$ *

$$+\left(-\cancel{\frac{1}{8}}+\cancel{\frac{1}{6}}\right)+\cdots$$

$$=\frac{1}{3}+\frac{1}{4}$$ *

$$=\boxed{\frac{7}{12}}$$ *

12. We rewrite the function as

$$\sqrt[5]{4}\left(1+\frac{x}{4}\right)^{1/5}.$$ *

By the binomial series with $k=\frac{1}{5}$, the first three terms are

$$\sqrt[5]{4}\left[1+\frac{1}{5}\left(\frac{x}{4}\right)+\frac{\frac{1}{5}\left(\frac{1}{5}-1\right)}{2!}\left(\frac{x}{4}\right)^2\right]=\boxed{\sqrt[5]{4}+\frac{\sqrt[5]{4}}{20}x-\frac{\sqrt[5]{4}}{200}x^2}$$ ****

(One point is awarded per correct term, and another point is awarded for a correct final answer.)

13. By the Alternating Series Error Bound, the error bound in S_N is

$$\left|\frac{(-1)^{N+1}}{(N+1+4)^3}\right|=\frac{1}{(N+5)^3}.$$ **

Now we set the expression to be less than 0.001 :

$$\frac{1}{(N+5)^3} < 0.001$$ *

$$(N+5)^3 > 1000$$

$$N+5 > 10$$

$$N > 5$$ *

The smallest integer is therefore

$$N = \boxed{6}$$ *

14. Differentiating gives

$$f'(x) = -2\sin 2x,$$

$$f''(x) = -4\cos 2x,$$

$$f'''(x) = 8\sin 2x.$$

The zeroth-degree term is

$$f\left(\frac{\pi}{3}\right) = -\frac{1}{2}.$$ *

The first-degree term is

$$f'\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right) = -\sqrt{3}\left(x-\frac{\pi}{3}\right).$$ *

The second-degree term is

$$\frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\frac{\pi}{3}\right)^2 = \left(x-\frac{\pi}{3}\right)^2.$$ *

The third-degree term is

$$\frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x-\frac{\pi}{3}\right)^3 = \frac{2\sqrt{3}}{3}\left(x-\frac{\pi}{3}\right)^3.$$ *

Hence, the Taylor polynomial is

$$T_3(x) = \boxed{-\frac{1}{2}-\sqrt{3}\left(x-\frac{\pi}{3}\right)+\left(x-\frac{\pi}{3}\right)^2+\frac{2\sqrt{3}}{3}\left(x-\frac{\pi}{3}\right)^3}$$ *

15. A Maclaurin series for $\ln(1+x)$ is

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \qquad *$$

for $-1 < x \leqslant 1$. Thus, replacing x with $x/2$ gives

$$\ln\left(1+\frac{x}{2}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\left(\frac{x}{2}\right)^n}{n} = \boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{2^n n}} \qquad **$$

The interval of convergence is $-1 < x/2 \leqslant 1$, or $\boxed{-2 < x \leqslant 2}$. **

16. *Conditions*. The function $f(x) = \dfrac{1}{(x+5)^3}$ is continuous, positive, and decreasing for $x \geqslant 5$. *
Thus, the Integral Test provides the error bound

$$R_5 \leqslant \int_5^{\infty} \frac{1}{(x+5)^3}\, \mathrm{d}x\,. \qquad *$$

The improper integral equals

$$\begin{aligned}
\lim_{t\to\infty} \int_5^t \frac{1}{(x+5)^3}\, \mathrm{d}x &= \lim_{t\to\infty} \left(-\frac{1}{2(x+5)^2}\right)\Bigg|_5^t \qquad * \\
&= \lim_{t\to\infty} \left(-\frac{1}{2(t+5)^2}\right) + \frac{1}{2(5+5)^2} \\
&= \boxed{\frac{1}{200}} \qquad **
\end{aligned}$$

17. We have

$$\begin{aligned} y' &= \frac{2x^3}{1-(-x^7)} \\ &= 2x^3 \sum_{n=0}^{\infty} (-x^7)^n \quad * \\ &= 2x^3 \sum_{n=0}^{\infty} (-1)^n x^{7n} \\ &= 2 \sum_{n=0}^{\infty} (-1)^n x^{7n+3}. \end{aligned}$$

Consequently,

$$\begin{aligned} y &= \int 2 \sum_{n=0}^{\infty} (-1)^n x^{7n+3}\,\mathrm{d}x \quad * \\ &= 2 \sum_{n=0}^{\infty} (-1)^n \int x^{7n+3}\,\mathrm{d}x \\ &= C + 2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+4}}{7n+4}. \quad * \end{aligned}$$

Substituting the initial condition $y(0)=5$ shows

$$\begin{aligned} y(0) = C + 2 \sum_{n=0}^{\infty} (-1)^n \frac{(0)^{7n+4}}{7n+4} &= 5 \quad * \\ C + 0 &= 5 \\ \implies C &= 5. \end{aligned}$$

Thus, the solution is

$$y = \boxed{5 + 2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+4}}{7n+4}} \quad *$$

Long Questions (15 points each)

18. (a) Note that $-1 \leqslant \frac{\cos n}{n} \leqslant 1$ (since $n \geqslant 1$). Adding 2 to each term gives

$$1 \leqslant 2+\frac{\cos n}{n} \leqslant 3\,.$$

Thus, the sequence is bounded. But cosine oscillates, so $\frac{\cos n}{n}$ (and therefore a_n) does not follow a uniform pattern of increasing or decreasing. Therefore, the answers are as follows:

$$\text{Bounded: } \boxed{\text{Yes}}$$ **

$$\text{Monotonic: } \boxed{\text{No}}$$ *

(b) Note that

$$-\frac{1}{n} \leqslant \frac{\cos n}{n} \leqslant \frac{1}{n}\,.$$ *

As $n \to \infty$, both $-\frac{1}{n}$ and $\frac{1}{n}$ approach 0. By the Squeeze Theorem,

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0\,.$$ *

Hence,

$$\lim_{n\to\infty} \left(2+\frac{\cos n}{n}\right) = 2+0 = \boxed{2}$$ **

(c) We have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{2n^2} = \frac{1}{2}\,.$$ *

By limit laws,

$$\lim_{n\to\infty} \left(\frac{2a_n}{b_n}\right) = \frac{2\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$$ *

$$= \frac{2(2)}{1/2}$$ *

$$= \boxed{8}$$ *

(d) • *Alternating*. The terms are alternating (starting negative at $n = 1$), so c_n must contain the factor $(-1)^n$.

- *Numerator*. The numerator doubles each iteration, starting at 1 when $n = 1$. Hence, c_n must contain 2^{n-1} in the numerator.
- *Denominator*. The denominator increases by 1 each time and begins at 4 when $n = 1$, so c_n must contain $(n-1)+4 = n+3$ in the denominator.

Hence,

$$c_n = \boxed{(-1)^n \frac{2^{n-1}}{n+3}}$$ ****

19. (a) Applying the Ratio Test, we have

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(n+1)+4} x^{n+1}}{\frac{(-1)^n}{n+4} x^n} \right| \quad * \\ &= \lim_{n\to\infty} \left| \frac{n+4}{n+5} x \right| \\ &= |x| \lim_{n\to\infty} \left(\frac{n+4}{n+5} \right) \\ &= |x| \,. \quad * \end{aligned}$$

The series converges absolutely when $L < 1$, so we have $|x| < 1$ or

$$-1 < x < 1 \,.$$ *

Next, we check the endpoints. When $x = -1$ the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+4} (-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n+4} = \sum_{n=0}^{\infty} \frac{1}{n+4} \,,$$

which diverges because it is the Harmonic series. In addition, when $x = 1$ the series becomes *

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+4} (1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+4} \,,$$

which converges by the Alternating Series Test because $\frac{1}{n+4}$ decreases to 0 as $n \to \infty$. Hence, the * interval of convergence is

$$\boxed{-1 < x \leqslant 1}$$ *

(b) Replacing x with $2x$ in the power series gives

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n+4}(2x)^n}$$ *

The interval of convergence becomes $-1 < 2x \leqslant 1$, or

$$\boxed{-\frac{1}{2} < x \leqslant \frac{1}{2}}$$ **

(c) Differentiating the series gives

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n+4} nx^{n-1}} \quad \text{or} \quad \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n+4} nx^{n-1}}$$ **

(d) We have

$$\int_0^x f(t)\,\mathrm{d}t = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{n+4} t^n\,\mathrm{d}t$$ *

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+4} \int_0^x t^n\,\mathrm{d}t$$ *

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+4} \frac{t^{n+1}}{n+1}\bigg|_0^x$$ *

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n+4} \frac{x^{n+1}}{n+1}}$$ *

20. (a) Applying the Ratio Test, we first find

$$L = \lim_{n\to\infty}\left|\frac{(-1)^n \dfrac{4^{n+1}}{n}(x-5)^{n+1}}{(-1)^{n-1}\dfrac{4^n}{n}(x-5)^n}\right| \quad *$$

$$= \lim_{n\to\infty} |4(x-5)|$$

$$= 4|x-5|. \quad *$$

Convergence occurs when $L<1$, so we have $4|x-5|<1$, or $|x-5|<\frac{1}{4}$. As a result,

$$R = \boxed{\frac{1}{4}} \quad *$$

(b) The third-degree Taylor polynomial is

$$T_3(x) = \boxed{4(x-5)-8(x-5)^2+\frac{64}{3}(x-5)^3} \quad *$$

Then

$$f(6) \approx T_3(6)$$

$$= 4(6-5)-8(6-5)^2+\frac{64}{3}(6-5)^3$$

$$= \boxed{\frac{52}{3}} \quad *$$

(c) The magnitude of the maximum error is, by Taylor's Remainder Theorem,

$$|R_4(6)| \leqslant \frac{M}{4!}(6-5)^4 \quad **$$

$$= \frac{40}{4!}(1)^4$$

$$= \boxed{\frac{5}{3}} \quad *$$

(d) By the definition of a Taylor Series, the nth degree term is

$$\frac{f^{(n)}(c)}{n!}(x-c)^n.$$

From part (b), the first-degree term is $4(x-5)$; then

$$\frac{f'(5)}{1!}(x-5)^1 = 4(x-5)$$

$$\implies f'(5) = \boxed{4}$$ *

Similarly, the second-degree term is $-8(x-5)^2$, so

$$\frac{f''(5)}{2!}(x-5)^2 = -8(x-5)^2$$

$$\implies f''(5) = \boxed{-16}$$ **

(e) The Maclaurin series for sine is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$ *

Multiplying the third-degree Maclaurin polynomials of f and sine shows

$$f(x)\sin x \approx \left(4 - \frac{x^2}{5} + 2x^3\right)\left(x - \frac{x^3}{6}\right)$$ *

$$= 4x - \frac{4x^3}{6} - \frac{x^3}{5} + \frac{x^5}{30} + 2x^4 - \frac{2x^6}{6}$$ *

$$= 4x - \frac{13x^3}{15} + 2x^4 + \frac{x^5}{30} - \frac{x^6}{3}.$$

Hence, the third-degree Maclaurin polynomial is

$$\boxed{4x - \frac{13x^3}{15}}$$ *